Double Ore Extensions versus Iterated Ore Extensions

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Abstract

Motivated by the construction of new examples of Artin-Schelter regular algebras of global dimension four, J.J.Zhang and J.Zhang (2008) introduced an algebra extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ of A, which they called a double Ore extension. This construction seems to be similar to that of a two-step iterated Ore extension over A. The aim of this paper is to describe those double Ore extensions which can be presented as iterated Ore extensions of the form $A[y_1; \sigma_1, \delta_1][y_2; \sigma_2, \delta_2]$. We also give partial answers to some questions posed in Zhang and Zhang (2008).

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Introduction

In 2008, J.J. Zhang and J. Zhang introduced a new construction for extending a given algebra A, by simultaneously adjoining two generators, y_1 and y_2. This construction resembles that of an Ore extension, and it was indeed called a double Ore extension (or double extension, for short). It should be noted that there are no inclusions between the classes of all double extensions of an algebra A and of all length two iterated Ore extensions of A. The aim of this paper is to describe the common part of these two classes of extensions of A.

In Section 1 we parallel the constructions of double extensions and Ore extensions, taking the opportunity to correct some typos which occurred in Zhang and Zhang (2008, p. 2674) and again in Zhang and Zhang (2009, p. 379), concerning the relations that the data of a double extension must satisfy. In Section 2 we present our main results, Theorems 2.2 and 2.4, which offer necessary and sufficient conditions for a double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ to be presented as iterated Ore extensions of the form $A[y_1; \sigma_1, \delta_1][y_2; \sigma_2, \delta_2]$ or $A[y_2; \sigma_2, \delta_2][y_1; \sigma_1 \delta_1]$. These, along with Lemma 2.3, give necessary conditions for a double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ to be presented as an iterated Ore extension $A[x_1; \sigma_1, \delta_1][x_2; \sigma_2, \delta_2]$, with x_1 and x_2 a basis of the vector space spanned by y_1 and y_2.

In Zhang and Zhang (2009), the authors pursue the study of Artin-Schelter regular algebras of global dimension four, by classifying certain types of double extensions and establishing some of their properties. So as to simplify their task, they develop criteria for a double extension, of the type they considered, to be an iterated Ore extension. This is obtained in Zhang and Zhang (2008, Proposition 3.6), which is a special case of Theorems 2.2 and 2.4 below.

We conclude with some applications and give partial answers to some questions posed in Zhang and Zhang (2008).

1 Double Ore Extensions

Throughout this paper, K denotes a field of arbitrary characteristic and K^* is its multiplicative group of units. For a K-algebra B, the algebra of n by m matrices with entries in B will be denoted by $M_{n\times m}(B)$.

Let A be a subalgebra of a K-algebra R and $x \in R$ be such that R is a free left A-module with basis $\{x^i\}_{i=0}^{\infty}$ and $xA \subseteq Ax + A$. Then, for any $a \in A$, there exist $\sigma(a), d(a) \in A$ such that $xa = \sigma(a)x + d(a)$. It is well known (cf. Cohn (1971)) that the above conditions imply that σ is an endomorphism of A and d is a σ-derivation of A, i.e., d is a K-linear map such that $d(ab) = \sigma(a)d(b) + d(a)b$, for all $a, b \in A$. Conversely, if an endomorphism σ and a σ-derivation d of a K-algebra A are given, then the multiplication in A and the condition $xa = \sigma(a)x + d(a)$ induce a structure of an associative K-algebra on the free left A-module with basis $\{x^i\}_{i=0}^{\infty}$. This extension is called an Ore extension and is denoted by $A[x; \sigma, d]$. One can easily check that the Ore extension $A[x; \sigma, d]$ is a free right A-module with basis $\{x^i\}_{i=0}^{\infty}$ if and only if σ is an automorphism of A if and only if σ is injective and $xA + A = Ax + A$.

We will now recall the definition of a double extension, as given in Zhang and Zhang (2008).

Definition 1.1. *Let A be a subalgebra of a K-algebra B. Then:*

(a) *B is called a right double extension of A if:*

  (i) *B is generated by A and two new variables y_1 and y_2;*

  (ii) *y_1 and y_2 satisfy the relation*

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1y_1 + \tau_2y_2 + \tau_0, \tag{1.I}$$

  for some $p_{12}, p_{11} \in K$ and $\tau_1, \tau_2, \tau_0 \in A$;

  (iii) *B is a free left A-module with basis $\{y_1^iy_2^j : i, j \geq 0\}$;*

  (iv) *$y_1A + y_2A + A \subseteq Ay_1 + Ay_2 + A$.*

(b) *A right double extension B of A is called a double extension if:*

  (i) *$p_{12} \neq 0$;*

  (ii) *B is a free right A-module with basis $\{y_2^iy_1^j : i, j \geq 0\}$;*

  (iii) *$y_1A + y_2A + A = Ay_1 + Ay_2 + A$.*

Condition (a)(*iv*) from the above definition is equivalent to the existence of two maps

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} : A \to M_{2\times 2}(A) \quad \text{and} \quad \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} : A \to M_{2\times 1}(A),$$

such that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} a = \sigma(a) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(a), \text{ for all } a \in A. \tag{1.II}$$

In case B is a right double extension of A, we will write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, where $P = \{p_{12}, p_{11}\} \subseteq K$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq A$ and σ, δ are as above. The set P is called a *parameter* and τ a *tail*.

Suppose $A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension. Then, it is clear that all maps σ_{ij} and δ_i are endomorphisms of the K-vector space A. In Zhang and Zhang (2008, Lemma 1.7) the authors showed that σ must be a homomorphism of algebras and δ a σ-derivation, in the sense that δ is K-linear and satisfies $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$, for all $a, b \in A$. One can easily check that, if the matrix $\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is triangular, then both σ_{11} and σ_{22} are algebra homomorphisms.

It is known that a map $d\colon A \to A$ is a σ-derivation, where σ is an endomorphism of A, if and only if the map from A to $M_{2\times 2}(A)$ sending a onto $\begin{bmatrix} \sigma(a) & d(a) \\ 0 & a \end{bmatrix}$ is a homomorphism of algebras. This, in particular, implies that for any algebra endomorphism σ of $K[x]$ and any polynomial $w \in K[x]$, there exists a (unique) σ-derivation d of $K[x]$ such that $d(x) = w$.

Let us observe that if $\tau \subseteq K$, then the subalgebra of $A_P[y_1, y_2; \sigma, \delta, \tau]$ generated by y_1 and y_2 is the double extension $K_P[y_1, y_2; \sigma', \delta', \tau]$, where $\sigma' = \sigma\mid_K$ is the canonical embedding of K in $M_{2\times 2}(K)$ and $\delta' = \delta\mid_K = 0$ is the zero map. The following proposition shows that the latter is always an iterated Ore extension.

Proposition 1.2. *Let $B = K_P[y_1, y_2; \sigma', \delta', \tau]$. Then $B \simeq K[x_1][x_2; \sigma_2, d_2]$ is an iterated Ore extension, where σ_2 is the algebra endomorphism of the polynomial ring $K[x_1]$ defined by $\sigma_2(x_1) = p_{12}x_1 + \tau_2$ and d_2 is the σ_2-derivation of $K[x_1]$ given by $d_2(x_1) = p_{11}x_1^2 + \tau_1 x_1 + \tau_0$. Moreover, B is a double extension of K if and only if $p_{12} \neq 0$.*

Proof. The preceding remarks guarantee the existence (and uniqueness) of the σ_2-derivation d_2. Thus, the iterated Ore extension $K[x_1][x_2; \sigma_2, d_2]$ can be considered. It is routine to check that $x_2x_1 = p_{12}x_1x_2 + p_{11}x_1^2 + \tau_1x_1 + \tau_2x_2 + \tau_0$ holds in $K[x_1][x_2; \sigma_2, d_2]$. This means that there is an algebra homomorphism from $K[x_1][x_2; \sigma_2, \delta_2]$ onto B mapping x_i to y_i, $i = 1, 2$. Since $\{x_1^i x_2^j \mid i, j \geq 0\}$ and $\{y_1^i y_2^j \mid i, j \geq 0\}$ are bases of $K[x_1][x_2; \sigma_2, d_2]$ and B over K, respectively, the homomorphism is an isomorphism.

If $p_{12} \neq 0$, then σ_2 is an automorphism of $K[x_1]$. This implies that the set $\{x_2^i x_1^j \mid i, j \geq 0\}$ is a basis of $K[x_1][x_2; \sigma_2, d_2]$ as a (right) K-vector space, and thus the same is true for $\{y_2^i y_1^j \mid i, j \geq 0\}$ and B. Hence, B is a double extension. □

Remark 1.3. *Let $C = K[y_1][y_2; \sigma_2, d_2]$ be as in Proposition 1.2. Then, for any K-algebra A, we have:*

$$A \otimes_K C = A_P[y_1, y_2; \sigma, \delta, \tau],$$

where $\sigma = \begin{bmatrix} id_A & 0 \\ 0 & id_A \end{bmatrix}$, $\delta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P = \{p_{12}, p_{11}\}$ and $\tau = \{\tau_0, \tau_1, \tau_2\}$ are as in Proposition 1.2.

Proposition 1.4. *Given $P = \{p_{12}, p_{11}\} \subseteq K$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq K$, $\sigma\colon A \to M_{2\times 2}(A)$ an algebra homomorphism and $\delta\colon A \to M_{2\times 1}(A)$ a σ-derivation, let $C = K[y_1][y_2; \sigma_2, d_2]$ be as in Proposition 1.2. Then, the following conditions are equivalent:*

(a) *the right double extension* $A_P[y_1, y_2; \sigma, \delta, \tau]$ *exists;*

(b) *one can extend the multiplications from* A *and* C *to a multiplication in the vector space* $A \otimes_K C$, *satisfying* $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} a = \sigma(a) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \delta(a)$, *for all* $a \in A$.

Proof. The remark just before Proposition 1.2 implies that for any sets P and τ of data, with $\tau \subseteq K$, the iterated Ore extension C exists. Now it is easy to complete the proof by using Proposition 1.2 and the definition of a right double extension. $\square$

Using Bergman's diamond lemma (Bergman (1978)), Zhang and Zhang gave a universal construction for a right double extension. Unfortunately, there are three small typos in the relations (R3.4)–(R3.6) appearing in Zhang and Zhang (2008, p. 2674). These come originally from analogous typos in Zhang and Zhang (2008, Lemma 1.10(a) and Equation (E1.10.3)). For the convenience of the reader, we re-write relations (R3.1)–(R3.6) of Zhang and Zhang (2008) as relations (1.III)–(1.VIII) below, with the corrected typos underlined.

Proposition 1.5. *(Zhang and Zhand, 2008, Lemma 1.10, Proposition 1.11) Given a* K*-algebra* A, *let* σ *be a homomorphism from* A *to* $M_{2\times 2}(A)$, δ *a* σ*-derivation from* A *to* $M_{2\times 1}(A)$, $P = \{p_{12}, p_{11}\}$ *a set of elements of* K *and* $\tau = \{\tau_0, \tau_1, \tau_2\}$ *a set of elements of* A. *Then, the associative* K*-algebra* B *generated by* A, y_1 *and* y_2, *subject to the relations (1.I) and (1.II), is a right double extension if and only if the maps* σ_{ij} *and* ρ_k, $i \in \{1,2\}$, $j,k \in \{0,1,2\}$, *satisfy the six relations (1.III)–(1.VIII) below, where* $\sigma_{i0} = \delta_i$ *and* ρ_k *is a* <u>*right*</u> *multiplication by* τ_k.

$$\sigma_{21}\sigma_{11} + p_{11}\sigma_{22}\sigma_{11} = p_{11}\sigma_{11}^2 + p_{11}^2\sigma_{12}\sigma_{11} + p_{12}\sigma_{11}\sigma_{21} + p_{11}p_{12}\sigma_{12}\sigma_{21} \tag{1.III}$$

$$\sigma_{21}\sigma_{12} + p_{12}\sigma_{22}\sigma_{11} = p_{11}\sigma_{11}\sigma_{12} + p_{11}p_{12}\sigma_{12}\sigma_{11} + p_{12}\sigma_{11}\sigma_{22} + p_{12}^2\sigma_{12}\sigma_{21} \tag{1.IV}$$

$$\sigma_{22}\sigma_{12} = p_{11}\sigma_{12}^2 + p_{12}\sigma_{12}\sigma_{22} \tag{1.V}$$

$$\begin{aligned}\sigma_{20}\sigma_{11} + \sigma_{21}\sigma_{10} + \underline{\rho_1\sigma_{22}\sigma_{11}} &= p_{11}\left(\sigma_{10}\sigma_{11} + \sigma_{11}\sigma_{10} + \tau_1\sigma_{12}\sigma_{11}\right) \\ &\quad + p_{12}\left(\sigma_{10}\sigma_{21} + \sigma_{11}\sigma_{20} + \tau_1\sigma_{12}\sigma_{21}\right) + \tau_1\sigma_{11} + \tau_2\sigma_{21}\end{aligned} \tag{1.VI}$$

$$\begin{aligned}\sigma_{20}\sigma_{12} + \sigma_{22}\sigma_{10} + \underline{\rho_2\sigma_{22}\sigma_{11}} &= p_{11}\left(\sigma_{10}\sigma_{12} + \sigma_{12}\sigma_{10} + \tau_2\sigma_{12}\sigma_{11}\right) \\ &\quad + p_{12}\left(\sigma_{10}\sigma_{22} + \sigma_{12}\sigma_{20} + \tau_2\sigma_{12}\sigma_{21}\right) + \tau_1\sigma_{12} + \tau_2\sigma_{22}\end{aligned} \tag{1.VII}$$

$$\begin{aligned}\sigma_{20}\sigma_{10} + \underline{\rho_0\sigma_{22}\sigma_{11}} &= p_{11}\left(\sigma_{10}^2 + \tau_0\sigma_{12}\sigma_{11}\right) \\ &\quad + p_{12}\left(\sigma_{10}\sigma_{20} + \tau_0\sigma_{12}\sigma_{21}\right) + \tau_1\sigma_{10} + \tau_2\sigma_{20} + \tau_0 id_A\end{aligned} \tag{1.VIII}$$

Remarks 1.6.

1. *Proposition 1.4 can be used to obtain a direct proof of Proposition 1.5, i.e., one which does not use Bergman's diamond lemma, provided that* $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq K$.

2. *Proposition 1.5 implies the uniqueness, up to isomorphism, of a right double extension of* A, *with given* σ, δ, P *and* τ, *provided such an extension exists. Indeed, assume* $\overline{B} = A_P[y_1, y_2; \sigma, \delta, \tau]$ *is a right double extension of* A. *Then, by Zhang and Zhang (2008, Lemmas 1.7 and 1.10(b)), the data* σ, δ, P *and* τ *satisfy the conditions of Proposition 1.5. Let* B *be as in this proposition. Then, there is an algebra homomorphism from* B *to* $\overline{B}$ *which restricts to the identity on* A *and maps* $y_i \in B$ *to the corresponding element* $y_i \in \overline{B}$, $i = 1, 2$. *Since* B *is a free left* A*-module with basis* $\{y_1^i y_2^j : i, j \geq 0\}$ *and the same holds for* $\overline{B}$, *this map is an isomorphism, thus proving uniqueness.*

As noticed in Zhang and Zhang (2008, Remark 1.4), by choosing a suitable basis of the vector space $Ky_1 + Ky_2$, we can prove:

Lemma 1.7. *Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension.*

(a) *If $p_{11} \neq 0$ and $p_{12} = 1$, then*

$$B \simeq A_{\{1,1\}} \left[\overline{y}_1, \overline{y}_2; \begin{bmatrix} \sigma_{11} & p_{11}\sigma_{12} \\ p_{11}^{-1}\sigma_{21} & \sigma_{22} \end{bmatrix}, \begin{bmatrix} p_{11}\delta_1 \\ \delta_2 \end{bmatrix}, \overline{\tau} \right]$$

where $\overline{\tau} = \{p_{11}\tau_0, \tau_1, p_{11}\tau_2\}$, $\overline{y}_1 = p_{11}y_1$ and $\overline{y}_2 = y_2$.

(b) *If $p_{12} \neq 1$, then*

$$B \simeq A_{\{p_{12},0\}} \left[\overline{y}_1, \overline{y}_2; \begin{bmatrix} \sigma_{11} - q\sigma_{12} & \sigma_{12} \\ \sigma_{21} + q(\sigma_{11} - \sigma_{22}) - q^2\sigma_{12} & \sigma_{22} + q\sigma_{12} \end{bmatrix}, \begin{bmatrix} \delta_1 \\ \delta_2 + q\delta_1 \end{bmatrix}, \overline{\tau} \right]$$

where $q = \frac{p_{11}}{p_{12}-1}$, $\overline{\tau} = \{\tau_0, \tau_1 - q\tau_2, \tau_2\}$, $\overline{y}_1 = y_1$ and $\overline{y}_2 = y_2 + qy_1$.

Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension and suppose that $p_{12} \neq 1$. Then, as observed above, by choosing adequate generators $\overline{y}_i$ and (possibly) modifying the data σ, δ, τ, one can assume that $p_{11} = 0$. Now suppose $\overline{B} = A_P[\overline{y}_1, \overline{y}_2; \overline{\sigma}, \overline{\delta}, \overline{\tau}]$ is a right double extension with $p_{11} = 0$. Then $\overline{B}$ has a natural filtration, given by setting $\deg A = 0$ and $\deg \overline{y}_1 = \deg \overline{y}_2 = 1$. One can check, in view of relations (1.I) and (1.II), that the associated graded algebra $\mathfrak{gr}(B)$ is isomorphic to $A_P[\overline{y}_1, \overline{y}_2; \overline{\sigma}, 0, \{0, 0, 0\}]$. The above shows that the following holds:

Corollary 1.8. *Suppose that $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of A, with $p_{12} \neq 1$. Then, there exists a filtration on B such that the associated graded algebra $D = \mathfrak{gr}(B)$ can be presented as follows: D is generated over A by indeterminates z_1, z_2; it is free as a left A-module with basis $\{z_1^i z_2^j : i, j \geq 0\}$; multiplication in D is given by multiplication in A and the conditions $z_2 z_1 = p_{12} z_1 z_2$ and $z_1 A + z_2 A \subseteq A z_1 + A z_2$, with $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} a = \overline{\sigma}(a) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, where $\overline{\sigma}$ is obtained from σ and $q = \frac{p_{11}}{p_{12}-1}$ as in Lemma 1.7(b).*

Furthermore, in case B is a double extension, then D is also free as a right A-module with basis $\{z_2^i z_1^j : i, j \geq 0\}$ and $z_1 A + z_2 A = A z_1 + A z_2$.

Suppose that $\mathcal{P}$ is a ring-theoretical property which passes from the associated graded algebra $\mathfrak{gr}(C)$ to the (filtered) algebra C. The above yields that, while investigating the lifting of property $\mathcal{P}$ from A to a right double extension B of A, one needs only consider two cases: $P = \{1, 1\}$ and $P = \{p_{12}, 0\}$, with $p_{12} \in K$.

2 Double extensions as iterated skew polynomial rings

In general, an iterated Ore extension of the form $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ is not a right double extension. In spite of this, one can check that if

$$\begin{aligned} \sigma_2(A) &\subseteq A, & \sigma_2(y_1) &= p_{12} y_1 + \tau_2, \\ d_2(A) &\subseteq A y_1 + A, & d_2(y_1) &= p_{11} y_1^2 + \tau_1 y_1 + \tau_0 \end{aligned}$$

where $p_{ij} \in K$ and $\tau_i \in A$, then the given iterated Ore extension $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ is indeed a right double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$, with $P = \{p_{12}, p_{11}\}$, $\tau = \{\tau_0, \tau_1, \tau_2\}$, $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_{21} & \sigma_2|_A \end{bmatrix}$ and $\delta = \begin{bmatrix} d_1 \\ \delta_2 \end{bmatrix}$, where $\sigma_{21}, \delta_2 \colon A \to A$ are defined by the condition $d_2(a) = \sigma_{21}(a)y_1 + \delta_2(a) \in Ay_1 + A$, for $a \in A$.

In our next theorem, we give necessary and sufficient conditions for $A_P[y_1, y_2; \sigma, \delta, \tau]$ to be an iterated Ore extension of the form $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$. By this we mean that we determine when the identity map on A extends to an algebra isomorphism from $A_P[y_1, y_2; \sigma, \delta, \tau]$ to $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ sending y_i to y_i, $i = 1, 2$. To proceed with this, we need the following:

Lemma 2.1. *Let $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ be an iterated Ore extension such that $\sigma_2(A) \subseteq A$ and $\sigma_2(y_1) = py_1 + q$, for some $p \in K^*$ and $q \in A$. Then, $\sigma_1\sigma_2(a) = \sigma_2\sigma_1(a)$, for all $a \in A$.*

Proof. Let $a \in A$. Applying σ_2 to the equality $y_1 a = \sigma_1(a)y_1 + d_1(a)$, we obtain $p\sigma_1\sigma_2(a)y_1 + pd_1\sigma_2(a) + q\sigma_2(a) = p\sigma_2\sigma_1(a)y_1 + \sigma_2\sigma_1(a)q + \sigma_2 d_1(a)$. Since $p \in K^*$, the thesis follows. $\square$

Theorem 2.2. *Let A, B be K-algebras such that B is an extension of A. Assume $P = \{p_{12}, p_{11}\} \subseteq K$, $\tau = \{\tau_0, \tau_1, \tau_2\} \subseteq A$, σ is an algebra homomorphism from A to $M_{2\times 2}(A)$ and δ is a σ-derivation from A to $M_{2\times 1}(A)$.*

(a) *The following conditions are equivalent:*

(i) *$B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of A which can be presented as an iterated Ore extension $A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$;*

(ii) *$B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a right double extension of A with $\sigma_{12} = 0$;*

(iii) *$B = A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ is an iterated Ore extension such that*

$$\sigma_2(A) \subseteq A, \quad \sigma_2(y_1) = p_{12}y_1 + \tau_2,$$
$$d_2(A) \subseteq Ay_1 + A, \quad d_2(y_1) = p_{11}y_1^2 + \tau_1 y_1 + \tau_0,$$

for some $p_{ij} \in K$ and $\tau_i \in A$. The maps σ, δ, σ_i and δ_i, $i = 1, 2$, are related by: $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ \sigma_{21} & \sigma_2|_A \end{bmatrix}$, $\delta(a) = \begin{bmatrix} d_1(a) \\ d_2(a) - \sigma_{21}(a)y_1 \end{bmatrix}$, for all $a \in A$.

(b) *If any of the statements from (a) holds, then B is a double extension of A if and only if $\sigma_1 = \sigma_{11}$ and $\sigma_2|_A = \sigma_{22}$ are automorphisms of A and $p_{12} \neq 0$.*

Proof. Notice that, if $B = A_P[y_1, y_2; \sigma, \delta, \tau]$, then $y_1 A \subseteq Ay_1 + A$ if and only if $\sigma_{12} = 0$. Therefore $(a)(i) \Rightarrow (a)(ii)$.

Suppose now that $(a)(ii)$ holds. Then $y_1 A \subseteq Ay_1 + A$. Hence, every element of the subalgebra $A[y_1]$ of B can be written in the form $a_n y_1^n + \ldots + a_0$, for suitable $n \geq 0$ and $a_i \in A$. Since B is a free left A-module with basis $\{y_1^i y_2^j\}_{i,j=0}^{\infty}$, the elements $\{y_1^i\}_{i=0}^{\infty}$ are A-independent, i.e., $A[y_1]$ is a free left A-module with that basis. Multiplication in $A[y_1]$ is given by multiplication in A and the condition $y_1 a = \sigma_1(a)y_1 + d_1(a)$, where $\sigma_1 = \sigma_{11}$ and $d_1 = \delta_1$. Thus, $A[y_1] = A[y_1; \sigma_1, d_1]$.

Since B is a right double extension of A, B is a free left $A[y_1; \sigma_1, d_1]$-module with basis $\{y_2^i\}_{i=0}^{\infty}$. Relation (1.I) can be re-written as

$$y_2 y_1 = (p_{12}y_1 + \tau_2)y_2 + p_{11}y_1^2 + \tau_1 y_1 + \tau_0 \qquad \text{(2.IX)}$$

and, by (1.II), we also have:

$$y_2A \subseteq Ay_2 + (Ay_1 + A). \tag{2.X}$$

Thus, by the above,

$$y_2A[y_1] \subseteq A[y_1]y_2 + A[y_1]. \tag{2.XI}$$

This means that $B = A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$, for some endomorphism σ_2 and some σ_2-derivation d_2 of $A[y_1; \sigma_1, d_1]$. Conditions (2.IX) and (2.X) imply that $\sigma_2(y_1) = p_{12}y_1 + \tau_2$, $\sigma_2(A) \subseteq A$, $d_2(A) \subseteq Ay_1 + A$ and $d_2(y_1) = p_{11}y_1^2 + \tau_1 y_1 + \tau_0$. By (1.II),

$$y_2a = \sigma_{22}(a)y_2 + \sigma_{21}(a)y_1 + \delta_2(a), \text{ for all } a \in A.$$

Thus, we also have $\sigma_2(a) = \sigma_{22}(a)$ and $d_2(a) = \sigma_{21}(a)y_1 + \delta_2(a)$, for all $a \in A$. Hence $(a)(iii)$ holds.

As observed at the beginning of this section, $(a)(iii) \Rightarrow (a)(i)$ holds, and the proof of (a) is completed.

Assume now that B is a double extension, with $\sigma_{12} = 0$. Then, by definition, $p_{12} \neq 0$. J.J. Zhang and J. Zhang introduced the determinant of σ, $\det(\sigma)\colon A \to A$, by setting $\det(\sigma) = -p_{11}\sigma_{12}\sigma_{11} + \sigma_{22}\sigma_{11} - p_{12}\sigma_{12}\sigma_{21}$, and showed (cf. Zhang and Zhang (2008, Lemma 1.9 and Proposition 2.1(a)(b))) that $\det(\sigma)$ is an automorphism of A, provided that B is a double extension of A. As $\sigma_{12} = 0$, this implies that $\det(\sigma) = \sigma_{22}\sigma_{11}$ is invertible in $\mathrm{End}_K(A)$. Notice that Lemma 2.1 and (a)(iii) above yield that σ_{11} and σ_{22} commute. Therefore, both σ_{11} and σ_{22} are automorphisms of A.

Conversely, suppose that $\sigma_1 = \sigma_{11}$, $\sigma_2|_A = \sigma_{22}$ are automorphism of A and $p_{12} \neq 0$. Since $\sigma_2(y_1) = p_{12}y_1 + \tau_2$, this implies that σ_2 is an automorphism of $A[y_1; \sigma_1, d_1]$. Hence, $\{y_2^i y_1^j : i, j \geq 0\}$ is a basis of $B = A[y_1; \sigma_1, d_1][y_2; \sigma_2, d_2]$ as a right A-module, i.e., B is a double extension of A. This completes the proof of (b). □

The following lemma gives a necessary and sufficient condition for the matrix corresponding to σ to be triangularizable, by choosing adequate generators of $A_P[y_1, y_2; \sigma, \delta, \tau]$ from $Ky_1 + Ky_2$, i.e., it gives a necessary condition for a right double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ to be presented as an iterated Ore extension over A.

Lemma 2.3. *Let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension, $k, l \in K$ and $0 \neq z = ky_1 + ly_2 \in B$. Then:*

$$zA \subseteq Az + A \qquad \textit{iff} \qquad kl\sigma_{11} + l^2\sigma_{21} = kl\sigma_{22} + k^2\sigma_{12}.$$

Proof. If either $k = 0$ or $l = 0$, then the identity above reduces to $\sigma_{21} = 0$ or $\sigma_{12} = 0$, accordingly. This gives the thesis in this case.

Suppose $k, l \in K^*$. Let $a \in A$. One can compute that

$$za = \left(\sigma_{11}(a) + \tfrac{l}{k}\sigma_{21}(a)\right) ky_1 + \left(\tfrac{k}{l}\sigma_{12}(a) + \sigma_{22}(a)\right) ly_2 + k\delta_1(a) + l\delta_2(a).$$

This yields the thesis. □

Suppose that the right double extension $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ can be presented as an iterated Ore extension of the form $A[y_2, \sigma_2', d_2'][y_1; \sigma_1', d_1']$. Then, we must have $\sigma_{21} = 0$, as $y_2A \subseteq Ay_2 + A$. Notice also that p_{12} has to be nonzero, as otherwise the quadratic relation (1.I) would imply that $\{y_1^i\}_{i=0}^{\infty}$ is not a free basis of B as a left $A[y_2; \sigma_2', d_2']$-module. Now,

relation (1.I) together with $y_1y_2 \in A[y_2,\sigma_2',d_2']y_1 + A[y_2,\sigma_2',d_2']$ imply that $p_{11}=0$. In this case, the quadratic relation (1.I) becomes

$$y_2y_1 = p_{12}y_1y_2 + \tau_1y_1 + \tau_2y_2 + \tau_0. \tag{2.XII}$$

Observe that, in any right double extension B satisfying relation (2.XII), the set $\{y_2^iy_1^j : i,j \geq 0\}$ still forms a basis of B as a left A-module. The fact that this relation is left-right symmetric implies that there is an isomorphism

$$B \simeq A_{\{p_{12}^{-1},0\}}\left[y_2,y_1;\begin{bmatrix}\sigma_{22} & \sigma_{21}\\ \sigma_{12} & \sigma_{11}\end{bmatrix},\begin{bmatrix}\delta_2\\ \delta_1\end{bmatrix},\{-p_{12}^{-1}\tau_0,-p_{12}^{-1}\tau_1,-p_{12}^{-1}\tau_2\}\right]$$

realized by interchanging the roles of y_1 and y_2.

The remarks above, together with Theorem 2.2, yield the following (cf. Zhang and Zhang (2009, Proposition 3.6)):

Theorem 2.4. *Let $B = A_P[y_1,y_2;\sigma,\delta,\tau]$ be a right double extension of the K-algebra A, where $P=\{p_{12},p_{11}\} \subseteq K$, $\tau=\{\tau_0,\tau_1,\tau_2\}\subseteq A$, $\sigma\colon A\to M_{2\times 2}(A)$ is an algebra homomorphism and $\delta\colon A\to M_{2\times 1}(A)$ is a σ-derivation. Then, B can be presented as an iterated Ore extension $A[y_2;\sigma_2',d_2'][y_1;\sigma_1',d_1']$ if and only if $\sigma_{21}=0$, $p_{12}\neq 0$ and $p_{11}=0$. In this case, B is a double extension if and only if $\sigma_2'=\sigma_{22}$ and $\sigma_1'|_A=\sigma_{11}$ are automorphisms of A.*

Let $\mathcal{P}$ denote one of the following ring-theoretical properties: being left (right) noetherian, being a domain, being prime, being semiprime left (right) noetherian, being semiprime left (right) Goldie. It is known that $\mathcal{P}$ lifts from a ring R to an Ore extension $R[x;\sigma,d]$, provided that σ is an automorphism of R (cf. Lam (1997), Matczuk (1995), McConnel and Robson (2001)). Thus, Theorems 2.2 and 2.4 yield the following partial positive answers to some of the questions posed in Zhang and Zhang (2008):

Corollary 2.5. *Suppose that the K-algebra A possesses the property $\mathcal{P}$. Then, the double extension $B = A_P[y_1,y_2;\sigma,\delta,\tau]$ also has the property $\mathcal{P}$, provided that either $\sigma_{12}=0$, or $\sigma_{21}=0$ and $p_{11}=0$.*

In Zhang and Zhang (2008), the authors asked whether primeness (resp. semiprimeness) lifts from an algebra A to its double extension $A_P[y_1,y_2;\sigma,\delta,\tau]$. It is known that, in general, semiprimeness does not lift from A to an Ore extension $A[y_1;\sigma_1,0]=A[y_1;\sigma_1]$, even if σ_1 is an automorphism. For such a non-semiprime extension we know, by Theorem 2.2, that $A[y_1;\sigma_1][y_2]$ is a double extension, which is clearly not semiprime. For a specific example, one can take $A=\prod_{i\in\mathbb{Z}}K_i$, where $K_i=K$ is a copy of the base field and σ_1 is the "right shifting" automorphism of A. Then $ay_1A[y_1;\sigma_1]ay_1=0$, for $a=(a_i)$ with $a_0=1$ and $a_i=0$ if $i\neq 0$, i.e., $A[y_1;\sigma_1]$ is not semiprime. On the other hand, semiprimeness does lift from the algebra A to an Ore extension $A[y_1;\sigma_1]$, provided that σ_1 is an automorphism and A is noetherian. The problem of determining whether semiprimeness lifts from A to $A_P[y_1,y_2;\sigma,\delta,\tau]$ when A is noetherian still remains open.

One of the examples of a double extension which appeared in Zhang and Zhang (2008) is the following:

Example 2.6. *Let $A=K[x]$ and fix $a,b,c\in K$ with $b\neq 0$. Let $\sigma\colon A\longrightarrow M_{2\times 2}(A)$ be the algebra homomorphism given by $\sigma(x)=\begin{bmatrix}0 & b^{-1}x\\ bx & 0\end{bmatrix}$ and $\delta\colon A\to M_{2\times 1}(A)$ be the*

σ-derivation determined by the condition $\delta(x) = \begin{bmatrix} cx^2 \\ -bcx^2 \end{bmatrix}$. *Then, the double extension* $B^2(a,b,c) = A_{\{-1,0\}}[y_1, y_2; \sigma, \delta, \{0,0,ax^2\}]$ *exists and it is the* K*-algebra generated by* x, y_1, y_2, *subject to the relations:*

$$y_2y_1 = -y_1y_2 + ax^2, \quad y_1x = b^{-1}xy_2 + cx^2, \quad y_2x = bxy_1 - bcx^2. \tag{2.XIII}$$

It was stated in Zhang and Zhang (2008) that if $a \neq 0$, then the algebra $B^2(a,b,c)$ cannot be presented as an iterated Ore extension over $K[x]$. The following proposition shows that this is not so, in case the characteristic of the base field K is 2.

Proposition 2.7. *Let* $a, b, c \in K$ *with* $b \neq 0$. *The algebra* $B^2 = B^2(a,b,c)$ *has the following properties:*

(a) *Suppose that* $\mathrm{char}(K) = 2$. *Then* B^2 *is the differential operator algebra* $B^2 = K[x,z][y_2; d]$, *where* d *is the derivation of* $K[x,z]$ *determined by* $d(x) = xz - bcx^2$ *and* $d(z) = abx^2$. *In particular,* B^2 *can be presented as an iterated Ore extension over* $A = K[x]$.

(b) B^2 *is a noetherian domain.*

Proof. Suppose that $\mathrm{char}(K) = 2$. Let us take $z = by_1 + y_2$. One can check, using (2.XIII), that $zx = xz$. In particular, $zA = Az$. Notice also that $zA + y_2A + A = Az + Ay_2 + A$, as $y_1A + y_2A + A = Ay_1 + Ay_2 + A$. Taking y_2 as the second generator of B^2 in the double extension, we have:

$$y_2z = -zy_2 + 2y_2^2 + abx^2 = zy_2 + abx^2, \tag{2.XIV}$$

given that $\mathrm{char}(K) = 2$. The above identity implies that B^2 can be also presented as a double extension $B^2 = A_{\{-1,0\}}[z, y_2; \sigma', \delta', \{0,0,abx^2\}]$, for suitable maps σ' and δ'. The condition $zA \subseteq Az + A$ means that $\sigma'_{12} = 0$. Thus, Theorem 2.2 implies that B^2 is an iterated Ore extension over $A = K[x]$. In fact, using (2.XIII) and the characteristic of K, one can check that

$$y_2x = -xy_2 + xz - bcx^2 = xy_2 + xz - bcx^2. \tag{2.XV}$$

Then, the identities $xz = zx$, (2.XIV) and (2.XV) imply that $B^2 = K[x,z][y_2; d]$, where d is a derivation of $K[x,z]$ as described above, i.e., $d(x) = xz - bcx^2$ and $d(z) = abx^2$. This proves (a).

For(b), let $\mathrm{char}(K)$ be arbitrary. Notice that, as $p_{11} = 0$, the algebra B^2 is filtered, as described in the paragraph preceding Corollary 1.8. The associated graded algebra, $\mathfrak{gr}(B^2)$, is generated by x, z_1 and z_2, subject to the relations (cf. (2.XIII)):

$$\begin{aligned} z_2z_1 &= -z_1z_2; \\ z_1x &= b^{-1}xz_2; \\ z_2x &= bxz_1. \end{aligned}$$

Thus, $\mathfrak{gr}(B^2)$ is the iterated Ore extension

$$K[z_1][z_2; \sigma_1][x; \sigma_2],$$

where $\sigma_1(z_1) = -z_1$, $\sigma_2(z_2) = bz_1$, $\sigma_2(z_1) = b^{-1}z_2$. Therefore, $\mathfrak{gr}(B^2)$ is a noetherian domain, which implies that so is B^2. □

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